On random sections of the cube

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Abstract

Let $f(j, k, n)$ denote the expected number of j-faces of a random k-section of the n-cube. A formula for $f(0, k, n)$ is presented, and for $j \geq 1$, a lower bound for $f(j, k, n)$ is derived, which implies a precise asymptotic formula for $f(n - m, n - l, n)$ when $1 \leq l < m$ are fixed integers and $n \to \infty$.

1   Introduction

The principal object in this paper is the expected number of j-dimensional faces (in short, j-faces) of a random k-dimensional central section (in short, k-section) of the n-cube $B_\infty^n = [-1, 1]^n$ in $\mathbb{R}^n$. We denote this number by $f(j, k, n)$. The normalized rotation invariant measure on the set $G_{n,k}$ of all k-dimensional subspaces of $\mathbb{R}^n$ provides the probabilistic framework.

Section 2 contains a calculation of the expected number of vertices of a random k- section of the n-cube. The result is:

$$f(0, k, n) = 2^k \binom{n}{k} \sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(t B_\infty^{n-k}) \, dt, \tag{1}$$

where γ_{n-k} denotes the $(n-k)$-dimensional Gaussian probability measure.

In §3 we derive a lower bound for $f(j, k, n)$ for every $1 \le j < k < n$. The main result is:

$$\frac{f(0, k-j, n)}{f(j, k, n)} < \sqrt{\frac{2}{\pi}} \left(\frac{j(k-j)}{n-k+j} \right)^{1/2} \int_0^\infty e^{-\frac{j(k-j)}{n-k+j}t^2/2} \gamma_j(tB_\infty^j) \, dt.$$

The lower bound for $f(j, k, n)$ derived from this inequality, combined with (1), leads in some cases to asymptotically best possible results. For example, in §3 we deduce from it the following asymptotic formula, for fixed integers $1 \le l < m$:

$$f(n-m, n-l, n) \backsim \frac{(2n)^{m-l}}{(m-l)!}, \quad \text{as} \ \ n \to \infty. \tag{2}$$

The notation $a_n \sim b_n$ means: $a_n/b_n \to 1$ as $n \to \infty$. (2) can be interpreted as follows: the probability that a random fixed-codimensional subspace of $\mathbb{R}^n$ intersects a fixed-codimensional face of the n-cube, tends to 1 as $n \to \infty$. The formula (2) itself follows also from the work of Affentranger and Schneider. (See remark 1 of section 3 below). In [1], they found a formula for the expected number $\mathbf{E}\left(f_j(\Pi_k B_1^n)\right)$ of j-faces of an orthogonal projection of an n-polytope P onto a k-dimensional random subspace. Formula (5) of [1] reads as follows:

$$\mathbf{E}\left(f_j(\Pi_k P)\right) = f_j(P) - 2 \sum_{s \ge 0} \sum_{F \in \mathcal{F}_j(P)} \sum_{G \in \mathcal{F}_{k+1+2s}(P)} \beta(F, G) \gamma(G, P). \tag{3}$$

Here $\mathcal{F}_j(P)$ denotes the set of k-faces of P, and $f_j(P) = \operatorname{card} \mathcal{F}_j(P)$. $\beta(F, G)$ denotes the internal angle ([7], p. 297) of the face G at its face F, and $\gamma(G, P)$ — the external angle ([7], p. 308) of P at its face G. It is shown in [1] that (3) implies that if $0 \le j < k$ are fixed integers, then as $n \to \infty$,

$$\mathbf{E}\left(f_j(\Pi_k T^n)\right) \sim \frac{2^k}{\sqrt{k}} \binom{k}{j+1} \beta(T^j, T^{k-1})(\pi \log n)^{(k-1)/2}. \tag{4}$$

Here T^n stands for the regular n-simplex.

In a very recent work, [4], Böröczky, Jr. and Henk showed that (3) implies the same asymptotic formula (4) also for $\mathbf{E}\left(f_j(\Pi_k B_1^n)\right)$, where B_1^n is

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the regular cross-polytope. In addition, they found an asymptotic formula for the internal angles $\beta(T^j, T^{k-1})$, when $k/j^2 \to \infty$. Therefore if j is fixed, k is much larger than j^2 and n much larger than k, then explicit estimates for $\mathbf{E}\left(f_j(\Pi_k B_1^n)\right)$ are available. See [4] for more details. Explicit asymptotic formulas for $\mathbf{E}\left(f_j(\Pi_k T^n)\right)$, were established independently by Vershik and Sporyshev ([9]), when j, k are both proportional to n and $n \to \infty$.

A simple duality argument shows that

$$\mathbf{E}\left(f_j(\Pi_k B_1^n)\right) = f(k - j - 1, k, n).$$

Choose $j = k - 1$ in (4). Applying the result for B_1^n, one has

$$f(0, k, n) = \mathbf{E}\left(f_{k-1}(\Pi_k B_1^n)\right) \sim \frac{2^k}{\sqrt{k}}(\pi \log n)^{(k-1)/2}, \qquad \text{as } n \to \infty. \qquad (5)$$

The last asymptotic formula follows also from (1). In fact, if $\{g_i\}_{i=1}^m$ are independent $N(0,1)$ (that is, with mean 0 and variance 1) Gaussian variables then $\gamma_m(tB_\infty^m)$ coincides with the probability of the event $\{\max_{1 \le i \le m} |g_i| \le t\}$. This probabilistic interpretation allows a straightforward evaluation of the asymptotic behavior of the integral in (1), when k is fixed and $n \to \infty$.

Formula (1) also yields information about $f(0, k, n)$ for k not necessarily fixed. For example, if $k = n - 1$, then the integral in (1) can be computed and the result is:

$$f(0, n - 1, n) = \frac{2^n n}{\pi} \arctan \frac{1}{\sqrt{n-1}} \sim \frac{2^n \sqrt{n}}{\pi}. \qquad (6)$$

Particular values of the last formula were computed numerically in [4]. (Table 2). For the expected number of vertices of random sections of fixed codimension, we have the following inequality, which is a consequence of (1).

$$f(0, n - d, n) \ge \binom{n}{d} 2^n \left(\frac{1}{\pi} \arctan \frac{1}{\sqrt{n-d}}\right)^d, \qquad (d \ge 1).$$

Equality holds for $d = 1$.

To obtain a lower bound for $f(j, k, n)$, it turns out that it is useful to know an estimate for the Gaussian measure of a cone generated by a section

of a face of a cube. In §3 we find such an estimate, by modifying K. Ball's calculation of the maximal volume of a cube–section, based on Brascamp-Lieb's inequality. ([2]).

Dvoretzky's theorem on almost Euclidean sections asserts that there exists a function $k(\varepsilon, n) \geq 1$, tending to infinity as $n \to \infty$ for each fixed $\varepsilon > 0$, such that if K is an n-dimensional centrally symmetric convex body (that is, a convex compact set in $\mathbb{R}^n$ with non-empty interior, satisfying $K = -K$), and $\varepsilon > 0$, then for each $1 \leq k \leq k(\varepsilon, n)$ there exists a k-dimensional subspace X, and a linear automorphism T of X for which

$$X \cap B_2^n \subset T(X \cap K) \subset (1 + \varepsilon)(X \cap B_2^n), \tag{7}$$

where B_2^n denotes the Euclidean unit ball. The proof of Dvoretzky's theorem in [5] shows that $k(\varepsilon, n) \geq c\varepsilon^2 |\log \varepsilon|^{-1} \log n$, for some absolute constant $c > 0$. That proof determined the best possible dependence of k on n. The dependence of k on ε was improved by Gordon [6], who discovered another proof of Dvoretzky's theorem with $k(\varepsilon, n) \geq c\varepsilon^2 \log n$. Both proofs are probabilistic; they show that not only there exist almost Euclidean sections, but actually most sections are such. More precisely, if X is a random subspace whose dimension does not exceed $k(\varepsilon, n)$, then the probability that the section $X \cap K$ is $(1+\varepsilon)$-Euclidean (common terminology for expressing that (7) holds), tends to 1 as $n \to \infty$. These facts motivate an investigation of the random f-vector $\{f(j, k, n)\}_{j=0}^{k-1}$, especially since it is well known that every k-dimensional symmetric polytope that has $2n$ facets is affinely equivalent to a k-section of an n-cube.

2 Vertices

Let $G_{n,k}$ denote the set of k-dimensional subspaces of $\mathbb{R}^n$. We will denote its normalized rotation invariant measure by "Prob". Recall that this measure is related to the normalized Haar measure H of the orthogonal group $O(n)$

by the equality

$$\text{Prob}\,\{X \in B\} = H\{g \in \mathrm{O}(n) : g[e_i]_{i=1}^k \in B\},$$

where B is a Borel subset of $G_{n,k}$ and $[e_i]_{i=1}^k$ is the k-dimensional sub-space spanned by the first k unit vectors in $\mathbb{R}^n$. Fix $X \in G_{n,k}$. For each $0 \leq j \leq k - 1$, the set of j-faces of the polytope $X \cap B_\infty^n$ coincides with the set of intersections of $(n - k + j)$-faces of B_∞^n with X. Every $(n - k + j)$-face of B_∞^n has the same probability to be intersected. Therefore if one particular $(n - k + j)$-face F_{n-k+j} is fixed, then the expected number of j-faces of the section $X \cap B_\infty^n$ is equal to:

$$2^{k-j} \binom{n}{k - j} \text{Prob}\,\{X \cap F_{n-k+j} \neq \emptyset\}.$$

Let $C(F_{n-k+j})$ denote the cone generated by F_{n-k+j}:

$$C(F_{n-k+j}) = \bigcup_{x \in F_{n-k+j}} \{tx : t \geq 0\}.$$

Put $C_1(F_{n-k+j}) = C(F_{n-k+j}) \cap \mathbb{S}^{n-1}$. For every subspace X,

$$X \cap F_{n-k+j} \neq \emptyset \iff (X \cap \mathbb{S}^{n-1}) \cap C_1(F_{n-k+j}) \neq \emptyset.$$

For $n = 0, 1, \dots$ we denote by σ_n the normalized rotation-invariant measure on the unit-sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The next lemma will prove useful for dealing with intersections of subsets of the sphere with random subspaces.

Lemma 2.1 *Let l, m, n be positive integers satisfying $l + m \geq n - 1$. Suppose that $A \subset \mathbb{S}^m$ and $B \subset \mathbb{S}^l$ are Borel subsets. Then for $p = l + m - n + 1$,*

$$\int_{\mathrm{O}(n)} \sigma_p(gB \cap A)\, dH(g) = \sigma_l(B)\, \sigma_m(A). \tag{8}$$

To prove the lemma one observes that for fixed A (resp. B) the integral defines an invariant measure on $\mathbb{S}^l$ (resp. $\mathbb{S}^m$); the conclusion follows from that.

Lemma 2.1 is now applied to $B = X \cap \mathbb{S}^{n-1}$, which we denote by $\mathbb{S}^{k-1}$, and to $A = C_1(F_{n-k+j})$. For $l = k - 1$ and $m = n - k + j$ equality (8) becomes:

$$\int_{O(n)} \sigma_j(g\mathbb{S}^{k-1} \cap A) \, dH(g) = \sigma_{n-k+j}(A). \tag{9}$$

We are ready to compute the expected number of vertices. The Gaussian measure in $\mathbb{R}^m$ whose density is $(2\pi)^{-m/2} \exp(-\sum_1^m x_i^2/2)$ is denoted by γ_m.

Proposition 2.2 *The expected number of vertices of a random k-dimensional central section of the n-cube is given by the formula*

$$f(0, k, n) = 2^k \binom{n}{k} \sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k}) \, dt.$$

Proof. For each $g \in O(n)$ we have

$$g\mathbb{S}^{k-1} \cap C_1(F_{n-k}) = (\mathrm{span}(g\mathbb{S}^{k-1}) \cap C(F_{n-k})) \cap \mathbb{S}^{n-1}.$$

For almost every g the intersection $\mathrm{span}(g\mathbb{S}^{k-1}) \cap C(F_{n-k})$ is either the origin itself, or else a ray emanating from the origin. Therefore the intersection $g\mathbb{S}^{k-1} \cap C_1(F_{n-k})$ is either empty or a singleton, for almost every g. Choose $j = 0$ in (9), with $A = C_1(F_{n-k})$. Since the measure σ_0 is concentrated on two points giving mass $1/2$ to each, we deduce from (9) that

$$\mathrm{Prob}\,\{X \cap F_{n-k} \neq \emptyset\} = 2\sigma_{n-k}(C_1(F_{n-k})). \tag{10}$$

To compute the r.h.s of (10), consider an $(n - k)$-dimensional cube of edge-length 1 inside $\mathbb{R}^{n-k+1}$, at a distance $\sqrt{k}$ from the origin, form the cone it generates, and compute the measure of its intersection with the sphere $\mathbb{S}^{n-k}$. Invoking polar coordinates we see that

$$\sigma_{n-k}(C_1(F_{n-k})) = \gamma_{n-k+1}(C(F_{n-k})).$$

By rotational symmetry of the Gaussian measure we may assume that F_{n-k} is specifically the set $\{x : |x_i| \leq 1, 1 \leq i \leq n - k, x_{n-k+1} = \sqrt{k}\}$. The

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intersection of the hyper-plane $\{x_{n-k+1} = t\}$ with $C(F_{n-k})$ is an $(n-k)$-dimensional cube of edge-length $\frac{t}{\sqrt{k}}$. Therefore by Fubini's theorem

$$
\begin{aligned}
\gamma_{n-k+1}(C(F_{n-k})) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} \gamma_{n-k}(\frac{t}{\sqrt{k}} B_\infty^{n-k}) \, dt \\
&= \sqrt{\frac{k}{2\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k}) \, dt.
\end{aligned}
$$

The last equality, together with (10), implies the desired formula. $\square$

The next lemma points out the precise asymptotic behavior of $f(0, k, n)$ when k is fixed and $n \to \infty$, and also that of $f(n - m, n - l, n)$, when l, m are fixed and $n \to \infty$. (To be used in §3.)

Lemma 2.3 *Suppose that $\{\alpha_n\}_{n=1}^\infty$ is a sequence of real numbers that has a positive limit α. Then as $n \to \infty$,*

$$
\int_0^\infty e^{-\alpha_n t^2/2} \gamma_n(tB_\infty^n) \, dt \sim \Gamma(\alpha) \frac{\pi^{\alpha/2}}{\sqrt{2}} \frac{(\log n)^{(\alpha_n - 1)/2}}{n^{\alpha_n}} \tag{11}
$$

where Γ is the Gamma function.

Proof. Let $F_n(t) = \text{Prob}\{\max_i |g_i| \le t\}$, where $g_1, \ldots, g_n$ are independent $N(0,1)$-Gaussian variables. We have

$$
\gamma_n(tB_\infty^n) = \left(\sqrt{\frac{2}{\pi}} \int_0^t e^{-x^2/2} \, dx \right)^n = F_n(t).
$$

For $n > 1$, put

$$
a_n = \frac{1}{\sqrt{2 \log n}}, \quad \text{and} \quad b_n = \sqrt{2 \log n} - \frac{\log(\pi \log n)}{2\sqrt{2 \log n}}.
$$

The well known tail approximation

$$
\sqrt{\frac{2}{\pi}} \int_t^\infty e^{-x^2/2} \, dx = \sqrt{\frac{2}{\pi}} \frac{1 + o(1)}{t} e^{-t^2/2} \quad \text{as } t \to \infty, \tag{12}
$$

combined with a simple calculation, implies that

$$
\lim_{n \to \infty} F_n(a_n x + b_n) = \exp(-e^{-x}), \quad \forall x \in \mathbb{R}. \tag{13}
$$

A change of variables gives:

$$\int_0^\infty e^{-\alpha_n t^2/2}\gamma_n(tB_\infty^n)\,dt = a_n \int_{-b_n/a_n}^\infty e^{-\alpha_n(a_n x+b_n)^2/2}F_n(a_n x+b_n)\,dx$$

$$= \frac{\pi^{\alpha_n/2}}{\sqrt{2}}\frac{(\log n)^{(\alpha_n-1)/2}}{n^{\alpha_n}}e^{-o(1)}\int_{-\infty}^\infty e^{-x^2\,o(1)}e^{-\alpha_n x(1-o(1))}F_n(a_n x+b_n)\chi_n(x)\,dx.$$

Here χ_n stands for the characteristic function of the interval $[-b_n/a_n,\infty)$. All four terms of the integrand in the last integral are non-negative for each x. For $x \geq 0$ and sufficiently large n we have $e^{-\alpha_n x(1-o(1))} < e^{-\alpha x/2}$, while the rest of the terms are majorized by 1. For $x < 0$ and sufficiently large n, we have $F_n(a_n x+b_n) < 2\exp(-e^{|x|})$ and $e^{-\alpha_n x(1-o(1))} < e^{2\alpha|x|}$. Thus in both cases if n is sufficiently large, the integrand is dominated by an integrable function. By (13), the integrand converges pointwise to the function $e^{-\alpha x}\exp(-e^{-x})$; Lebesgue's bounded convergence theorem can be applied:

$$\lim_{n\to\infty}\int_{-b_n/a_n} e^{-x^2 o(1)}e^{-\alpha_n x(1-o(1))}F_n(a_n x+b_n)\,dx = \int_{-\infty}^\infty e^{-\alpha x}\exp(-e^{-x})\,dx$$

$$= \Gamma(\alpha).$$

The proof of Lemma 2.3 is complete. $\qquad\square$

Taking $\alpha_n \equiv k$ in Lemma 2.3 and bearing in mind Proposition 2.2 re-proves the following result, which was mentioned in the introduction.

Corollary 2.4 *For fixed k,*

$$f(0,k,n) \sim \frac{2^k}{\sqrt{k}}(\pi\log n)^{(k-1)/2}, \qquad \text{as } n\to\infty.$$

We turn now to the case of fixed co-dimension. The next result is deduced from proposition 2.2.

Proposition 2.5 *For $d \geq 1$,*

$$f(0,n-d,n) \geq \binom{n}{d}2^n\left(\frac{1}{\pi}\arctan\frac{1}{\sqrt{n-d}}\right)^d, \qquad (d \geq 1).$$

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Equality holds for $d = 1$:

$$f(0, n - 1, n) = \frac{2^n n}{\pi} \arctan \frac{1}{\sqrt{n - 1}}. \tag{14}$$

Proof. Consider the probability measure $d\mu(t) = 2\sqrt{\frac{k}{\pi}} e^{-kt^2} dt$ on the half-line $[0, \infty)$. Put

$$\Phi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx.$$

Then

$$\gamma_{n-k}(tB_\infty^{n-k}) = \left(\sqrt{\frac{2}{\pi}} \int_0^t e^{-x^2/2} dx \right)^{n-k} = \Phi^{n-k}(t/\sqrt{2}).$$

Therefore

$$\sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k}) \, dt = \sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \Phi^{n-k}(t/\sqrt{2}) \, dt$$

$$= \int_0^\infty \Phi^{n-k}(t) \, d\mu(t) \tag{15}$$

$$\geq \left(\int_0^\infty \Phi(t) \, d\mu(t) \right)^{n-k}.$$

Elementary calculation shows that

$$\int_0^\infty e^{-kt^2} \Phi(t) \, dt = \frac{1}{\sqrt{\pi k}} \arctan \frac{1}{\sqrt{k}}.$$

A combination of (15) with proposition 2.2 gives the desired inequality, after a replacement of k by $n - d$. Observe that for $k = n - 1$ (that is, $d = 1$), there is equality in the inequality of (15). $\qquad\square$

Remarks

1. For $n = 3$ we get from (14): $f(0, 2, 3) = (24/\pi) \arctan \frac{1}{\sqrt{2}} \approx 4.7$. Therefore a random 2-section of the 3-cube is more likely to be a parallelogram than a hexagon.

2. Bárány and Lovász proved in [3] that (in particular) almost every k-section of the n-cube has at least 2^k vertices. Clearly this is a precise

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lower bound. For $k = n - 1$, our result shows that the expected value is asymptotically $\sqrt{n}/\pi$ times the minimal value.

3. The asymptotic behavior of the integral

$$\int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k}) \, dt$$

for fixed k and $n \to \infty$ was determined in [4] (following [8]), and was used to prove formula (4) of the introduction. See also [1]. The asymptotic result is basically a corollary of the classical tail approximation of a single $N(0,1)$-Gaussian variable. Our approach to the proof of Lemma 2.3 seems to simplify the analysis.

4. As was indicated in the introduction, we can choose $\varepsilon = \frac{c}{\sqrt{\log n}}$ for some constant $c > 0$, and then with high probability a random 2-section of the cube is $(1 + \frac{c}{\sqrt{\log n}})$-Euclidean. It is well known that among all centrally symmetric polygons having $2m$ vertices, the regular $2m$-gon minimizes the Banach-Mazur distance to the Euclidean disc; the minimal distance is $(\cos(\pi/2m))^{-1}$. Consequently with high probability we have

$$(\cos(\pi/2m))^{-1} < 1 + \frac{c}{\sqrt{\log n}}.$$

Hence most 2-sections of the n-cube have at least $C(\log n)^{1/4}$ vertices, for some positive constant C. By Corollary 2.2 (after a suitable rearrangement)

$$f(0, 2, n) = 2\sqrt{\pi} \mathbf{E} \left(\max_{1 \le i \le n} |g_i| \right),$$

which is of the order of magnitude of $\sqrt{\log n}$. Summarizing these observations, we conclude: a typical 2-section of the n-cube is $(1 + \frac{c}{\sqrt{\log n}})$- Euclidean, hence it cannot have too few vertices — it has at least $C(\log n)^{1/4}$ vertices with probability that tends to 1 as $n \to \infty$. It does not however tend to be a regular polygon, because the expected number of its vertices is too high for that.

3 Other faces

We now turn to the case $j \ge 1$, and prove the following result.

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Proposition 3.1 *For $j \geq 1$, the following inequality holds.*

$$\frac{f(0, k-j, n)}{f(j, k, n)} < \sqrt{\frac{2}{\pi}} \left(\frac{j(k-j)}{n-k+j}\right)^{1/2} \int_0^\infty e^{-\frac{j(k-j)}{n-k+j}t^2/2} \gamma_j(tB_\infty^j) \, dt.$$

The starting point in the proof of Proposition 3.1 is (9) of Lemma 2.1. Again, we choose $A = C_1(F_{n-k+j})$. The random variable $g \to \sigma_j(g\mathbb{S}^{k-1} \cap A)$, which is defined on $O(n)$, has values in $[0, 1]$. Hence

$$\int_{O(n)} \sigma_j(g\mathbb{S}^{k-1} \cap A) \, dH(g) = \int_0^1 H\{g : \sigma_j(g\mathbb{S}^{k-1} \cap A) \geq t\} \, dt. \tag{16}$$

The integrand is non-increasing, and

$$H\{g : \sigma_j(g\mathbb{S}^{k-1} \cap A) \geq 0\} = \mathrm{Prob}\{X \cap F_{n-k+j} \neq \emptyset\}, \tag{17}$$

because the event $\{g\mathbb{S}^{k-1} \cap A \neq \emptyset$ and $\sigma_j(g\mathbb{S}^{k-1} \cap A) = 0\}$ has Haar measure zero. Therefore by (9):

$$\sigma_{n-k+j}(A) \leq \mathrm{Prob}\{X \cap F_{n-k+j} \neq \emptyset\} \sup\{t : H\{g : \sigma_j(g\mathbb{S}^{k-1} \cap A) \geq t\} > 0\}$$
$$\leq \mathrm{Prob}\{X \cap F_{n-k+j} \neq \emptyset\} \sup\{\sigma_j(g\mathbb{S}^{k-1} \cap A) : g \in O(n)\}.$$

Let

$$t_{j,k,n} = \sup\{\sigma_j(g\mathbb{S}^{k-1} \cap A) : g \in O(n)\}.$$

By (9), (15) and (16) we get

$$\mathrm{Prob}\{X \cap F_{n-k+j} \neq \emptyset\} \geq \frac{\sigma_{n-k+j}(A)}{t_{j,k,n}}.$$

Hence by (10)

$$f(j, k, n) \geq 2^{k-j} \binom{n}{k-j} \frac{\sigma_{n-k+j}(A)}{t_{j,k,n}} = \frac{\frac{1}{2}f(0, k-j, n)}{t_{j,k,n}}.$$

We must bound $t_{j,k,n}$ from above. Since A is contained in a half-space, a trivial bound is $t_{j,k,n} \leq \frac{1}{2}$. In some cases this bound can be significantly improved. The main lemma in this section is the following.

Lemma 3.2 *If $1 \leq j < k < n$, then*

$$t_{j,k,n} \leq \frac{1}{\sqrt{2\pi}} \left(\frac{j(k-j)}{n-k+j}\right)^{1/2} \int_0^\infty e^{-\frac{j(k-j)}{n-k+j}t^2/2} \gamma_j(tB_\infty^j) \, dt.$$

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The next lemma will be used in the proof of Lemma 3.2.

Lemma 3.3 *Given a positive number $\tau > 0$, a j-dimensional subspace Y of $\mathbb{R}^m$ and a point $y_0 \in Y$, the following inequality holds.*

$$\gamma_j((Y \cap \tau B_\infty^m) - y_0) \le \gamma_j(\tau \sqrt{m/j} B_\infty^j). \tag{18}$$

Proof. Let Q denote the orthogonal projection onto $Y - y_0$. As usual, $\{e_i\}_{i=1}^m$ are the standard unit vectors in $\mathbb{R}^m$. Put $u_i = Qe_i/\|Qe_i\|$ if $Qe_i \ne 0$, and $u_i = 0$ otherwise; put $c_i = \|Qe_i\|^2$ and $\alpha_i = \langle y_0, e_i \rangle$ for $1 \le i \le m$. ($\langle \cdot, \cdot \rangle$ is the standard scalar product.) Then

$$
\begin{aligned}
Y \cap \tau B_\infty^m &= \{y \in Y : |\langle y, e_i \rangle| \le \tau \ \ \forall i\} \\
&= \{y \in Y : |\langle y - y_0, e_i \rangle + \langle y_0, e_i \rangle| \le \tau \ \ \forall i\} \\
&= \{y \in Y : \frac{-\alpha_i - \tau}{\sqrt{c_i}} \le \langle y - y_0, u_i \rangle \le \frac{-\alpha_i + \tau}{\sqrt{c_i}}\}.
\end{aligned}
$$

Therefore

$$(Y \cap \tau B_\infty^m) - y_0 = \{x \in Y - y_0 : \frac{-\alpha_i - \tau}{\sqrt{c_i}} \le \langle x, u_i \rangle \le \frac{-\alpha_i + \tau}{\sqrt{c_i}}\}$$

Now we can imitate K. Ball's argument from [2] concerning sections of maximal volume. Instead of the Lebesgue measure, we have to consider the Gaussian measure.

In $Y - y_0$, the identity operator can be written as $\sum_1^m c_i u_i \otimes u_i$. In particular,

$$\sum_{i=1}^m c_i = j, \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^m c_i \langle x, u_i \rangle^2, \quad \forall x \in Y - y_0.$$

Therefore the Gaussian measure in $Y - y_0$ is equal to

$$(2\pi)^{-j/2} \exp(-\sum_{i=1}^m c_i \langle x, u_i \rangle^2 / 2) \, dx.$$

Let χ_i denote the characteristic function of the interval $[\frac{-\alpha_i - \tau}{\sqrt{c_i}}, \frac{-\alpha_i + \tau}{\sqrt{c_i}}]$. Then, by the above,

$$\gamma_j((Y \cap \tau B_\infty^m) - y_0) = (2\pi)^{-j/2} \int_{Y - y_0} \Big(\prod_{i=1}^m \chi_i(\langle x, u_i \rangle) e^{-c_i \langle x, u_i \rangle^2 / 2} \Big) dx$$

$$= (2\pi)^{-j/2} \int_{Y - y_0} \prod_{i=1}^m (\chi_i(\langle x, u_i \rangle) e^{-\langle x, u_i \rangle^2 / 2})^{c_i} dx \qquad (19)$$

$$\leq (2\pi)^{-j/2} \prod_{i=1}^m \Big(\int_{(-\alpha_i - \tau)/\sqrt{c_i}}^{(-\alpha_i + \tau)/\sqrt{c_i}} e^{-s^2/2} ds \Big)^{c_i}.$$

The last inequality is a consequence of Brascamp-Lieb's inequality, which is stated in [2] as follows:

Lemma *Let $(u_i)_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$ and $(c_i)_1^m$ a sequence of positive numbers so that*

$$\sum_1^m c_i u_i \otimes u_i = I_n.$$

For each i, let $f_i : \mathbb{R} \to [0, \infty)$ be integrable. Then

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle u_i, x \rangle)^{c_i} dx \leq \prod_{i=1}^m \Big(\int_{\mathbb{R}} f_i \Big)^{c_i}.$$

The i'th integral in the product of (19) is not larger than $\int_{-\tau/\sqrt{c_i}}^{\tau/\sqrt{c_i}} e^{-s^2/2} ds$. Hence the last expression in (19) is bounded above by

$$(2\pi)^{-j/2} \prod_{i=1}^m \Big(2 \int_0^{\tau/\sqrt{c_i}} e^{-s^2/2} ds \Big)^{c_i},$$

which is maximized when all the c_i's are equal. Hence

$$\gamma_j((Y \cap \tau B_\infty^m) - y_0) \leq \Big(\sqrt{\frac{2}{\pi}} \int_0^{\tau \sqrt{m/j}} e^{-s^2/2} ds \Big)^j \qquad (20)$$

$$= \gamma_j(\tau \sqrt{m/j} B_\infty^j).$$

The proof of Lemma 3.3 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Lemma 3.2

For $g \in \mathrm{O}(n)$

$$\sigma_j(g\mathbb{S}^{k-1} \cap A) = \gamma_{j+1}\big(C(F_{n-k+j}) \cap \mathrm{span}(g\mathbb{S}^{k-1})\big)$$

$$= \gamma_{j+1}\big(C[F_{n-k+j} \cap \mathrm{span}(g\mathbb{S}^{k-1})]\big).$$

The second equality is a consequence of the identity $C(F_{n-k+j}) \cap X = C(F_{n-k+j} \cap X)$, which trivially holds for every subspace $X \subset \mathbb{R}^n$. Fix a subspace $X \in G_{n,k}$ for which the section $X \cap F_{n-k+j}$ is j-dimensional; almost every $X \in G_{n,k}$ has this property. Let C denote the $(j+1)$-dimensional cone generated by $X \cap F_{n-k+j}$; put $X_0 = \mathrm{span}C$. By M we denote the affine subspace spanned by $X \cap F_{n-k+j}$, and by d, its distance from the origin of X. The Gaussian measure of the cone C is computed as follows. Take the unit vector $\xi \in X_0$ which is orthogonal to M, and for which $d\xi \in M$. For $t > 0$, put $W_t = \{x \in X_0 : \langle x, \xi \rangle = t\}$. Observe that $C \cap W_t = (t/d)(X \cap F_{n-k+j})$. Let P denote the orthogonal projection from X_0 onto W_0. By Fubini's theorem:

$$\gamma_{j+1}(C) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} \gamma_j(P(C \cap W_t))\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} \gamma_j(P(t/d)(X \cap F_{n-k+j}))\, dt. \tag{21}$$

Our task is to estimate the expression $\gamma_j(P\tau(X \cap F_{n-k+j}))$ for every $\tau > 0$. We will need to discuss Gaussian measures in different subspaces. Whenever M is an m-dimensional subspace of $\mathbb{R}^n$ and $q \in M$, let $\mathbb{G}_{M,q}$ denote the measure $(2\pi)^{-m/2} \exp(-\|x-q\|^2/2)\, dx$. In case M is an m-dimensional linear subspace of $\mathbb{R}^n$ and $q = 0$ we shall simply write $\mathbb{G}_{M,0} = \gamma_m$. If T is an isometry of $\mathbb{R}^n$, then for every Borel subset $S \subset M$ we have

$$\mathbb{G}_{M,q}(S) = \mathbb{G}_{TM,Tq}(TS). \tag{22}$$

Let us momentarily assume that $\tau = 1$. Let q denote the nearest point of M to the origin of X. Both M and the range of the projection P are j-dimensional affine subspaces of X_0. We have

$$P(X \cap F_{n-k+j}) = (X \cap F_{n-k+j}) - q,$$

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hence by (22)

$$\mathbb{G}_{M,q}(X \cap F_{n-k+j}) = \gamma_j(P(X \cap F_{n-k+j})). \qquad (23)$$

Now let L denote the affine subspace spanned by F_{n-k+j}, whose origin O_L is taken as the center of the face F_{n-k+j}. (So if X passes through the center of F_{n-k+j}, then $q = O_L$.) M is also a j-dimensional affine subspace of L. By (22),

$$\mathbb{G}_{M,q}(X \cap F_{n-k+j}) = \mathbb{G}_{M-(q-O_L),O_L}\big((X \cap F_{n-k+j}) - (q - O_L)\big).$$

Applying the same argument for arbitrary $\tau > 0$ we conclude that

$$\gamma_j(P\tau(X \cap F_{n-k+j})) = \mathbb{G}_{\tau M - \tau(q-O_L),\tau O_L}\big(\tau(X \cap F_{n-k+j}) - \tau(q - O_L)\big). \qquad (24)$$

We may think of τL as $\mathbb{R}^{n-k+j}$, of τF_{n-k+j} as τB_∞^{n-k+j}, and of $\tau(X \cap F_{n-k+j})$ as an affine j-dimensional section of τB_∞^{n-k+j}. Thus for each $t > 0$ Lemma 3.3 can be used with $\tau = t/d$ and $m = n - k + j$. By the definition of d, we have $d \geq \sqrt{k-j}$. Combining (18),(21) and (24) we deduce that

$$\gamma_{j+1}(C) \leq \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} \gamma_j\left(t\left(\frac{n-k+j}{j(k-j)}\right)^{1/2} B_\infty^j\right) dt$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{j(k-j)}{n-k+j}\right)^{1/2} \int_0^\infty \exp\left(-\frac{j(k-j)}{n-k+j} t^2/2\right) \gamma_j(t B_\infty^j) \, dt.$$

The proof of lemma 3.2 and thus of proposition 3.1 is complete. $\qquad \square$

By using the asymptotic formulas of section 2, namely Lemma 2.3 and Corollary 2.4, we can now prove the following result, which shows that the lower bound for $f(j, k, n)$ derived from proposition 3.1 is, in some cases, asymptotically best possible.

Corollary 3.4 *For fixed integers $1 \leq l < m$,*

$$f(n - m, n - l, n) \backsim \frac{(2n)^{m-l}}{(m-l)!} \quad as \;\; n \to \infty. \qquad (25)$$

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Proof. Put $\alpha_n = (m - l)(n - m)/(n - m + l)$. By Proposition 3.1,

$$\frac{f(0, m - l, n)}{f(n - m, n - l, n)} < \sqrt{\frac{2\alpha_n}{\pi}} \int_0^\infty e^{-\alpha_n t^2/2} \gamma_{n-m}(tB_\infty^{n-m}) \, dt. \qquad (26)$$

Put $b_n = (\log(n - m))^{(\alpha_n - 1)/2}/(n - m)^{\alpha_n}$ and $c_n = (\log n)^{(m-l-1)/2}$. Let d_n denote the right hand side of (26), from which we get

$$f(n - m, n - l, n)\frac{b_n}{c_n} > \frac{f(0, m - l, n)}{c_n}\frac{b_n}{d_n}.$$

Since $\lim_{n\to\infty} \alpha_n = (m - l)$, Lemma 2.3 implies that

$$\lim_{n\to\infty} \frac{b_n}{d_n} = \frac{1}{\pi^{(m-l-1)/2}\Gamma(m - l)\sqrt{m - l}}.$$

Moreover, by Corollary 2.4,

$$\lim_{n\to\infty} \frac{f(0, m - l, n)}{c_n} = \frac{2^{m-l}\pi^{(m-l-1)/2}}{\sqrt{m - l}}.$$

Thus, the sequence $f(n - m, n - l, n)\frac{b_n}{c_n}$ is larger than a sequence that tends to $2^{m-l}/(m - l)!$ as n tends to infinity. On the other hand we have $f(n - m, n - l, n) < 2^{m-l}\binom{n}{m-l}$, so

$$f(n - m, n - l, n)\frac{b_n}{c_n} < 2^{m-l}\binom{n}{m - l}\frac{b_n}{c_n},$$

and since $b_n/c_n \backsim n^{l-m}$, the r.h.s here tends to $2^{m-l}/(m - l)!$. Consequently,

$$\lim_{n\to\infty} f(n - m, n - l, n)\frac{b_n}{c_n} = \frac{2^{m-l}}{(m - l)!}.$$

The required asymptotic formula follows immediately. The proof of Corollary 3.4 is complete. $\qquad\square$

Remarks

1. The previous corollary implies that the number of $(n - m)$-faces of a random $(n - l)$-section of the n-cube tends to concentrate near the value $2^{m-l}\binom{n}{m-l}$, which bounds it from above. So for example, a typical 1-co-dimensional section of the n-cube will have $2n - o(n)$ facets as $n \to \infty$. This

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result can also be deduced from the identity (3). Indeed, by duality we have $f(n-m, n-l, n) = \mathbf{E}\left(f_{m-l-1}(\Pi_{n-l}(B_1^n))\right)$, and replacing T^n by B_1^n in the proof of Theorem 2 in [1], (the details of this replacement appear in [4]; see the proof of Theorem 1.1 there) we get the previous corollary.

2. According to a remark made in [4], the number $f(j, k, n)$ is equal to the expected number of $(k-j-1)$-faces of the convex hull of $\pm G_1, \ldots, \pm G_n$, where the G_i's are independent copies of a k-dimensional Gaussian vector. Hence, the results for $f(0, k, n)$ can be interpreted as results for the expected number of facets of the convex hull of $\{\pm G_i\}_1^n$ in $\mathbb{R}^k$. For example, we can translate the first remark at the end of section 2 to the following statement:

If 3 points in the plane are chosen at random, then their symmetric convex hull is more likely to be a parallelogram than a hexagon.

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